Math 141
Joseph C Foster
Summer 2017
Final Exam

Name: Solutions

June 24th, 2017
Time Limit: 150 minutes

This exam contains 12 pages (including this cover page) and 21 questions. The total number of marks is 100. You have 150 minutes to complete the exam.

Read each question carefully. When specified, you must show **all** *necessary* work to receive full credit.

Try not to leave any questions blank. This is your chance to show me what you know. If you are stumped, write down any definitions you think you may need, or write out the steps you want to take, even if the calculation is too hard. These things will bag you partial credit at the very least. I cannot give marks for a blank page.

If you need scrap paper, raise your hand. There is a detachable formula sheet on the last page.

No calculator/phone/smartwatch allowed under any circumstances. Place these items in your bag, out of reach. Cheating of any kind will not be tolerated and will result in a grade of zero.

Question	Marks	Score	Question	Marks	Score
1	3		12	4	
2	3		13	5	
3	3		14	5	
4	3		15	6	
5	3		16	6	
6	3		17	6	
7	3		18	7	
8	3		19	7	
9	3		20	10	
10	3		21	10	
11	4		Total	100	

1. (3 marks) True or False: If a function is continuous at a point c, then it is differentiable at the point c.

   A. True B. False

2. (3 marks) True or False: $\lim_{x\to 0} \frac{\sin(4x)}{6x} = \frac{2}{3}$.

   A. True B. False

3. (3 marks) True or False: If $y = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$, then $y' = \frac{f(x)g'(x) - f'(x)g(x)}{[g(x)]^2}$.

   A. True B. False

4. (3 marks) Fill in the blank: Let $f(x)$ be a function. The limit definition of the derivative is

   $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ or $\lim_{z\to x} \frac{f(z)-f(x)}{z-x}$.

   (*Either of the two definitions we learnt is fine*)

5. (3 marks) Fill in the blanks: Let $f(x)$ be a function. Then c is a critical point of f if

   $f'(c)=0$ *or* $f'(c)$ DNE.

6. (3 marks) Fill in the blanks: Let $f(x)$ be a twice differentiable function. Let $c \in \mathbb{R}$ be a point such that $f'(c) = 0$. Then;

   - If $f''(c) < 0$, $f(c)$ is a local maximum.
   - If $f''(c) > 0$, $f(c)$ is a local minimum.
   - If $f''(c) = 0$, we know nothing.

For questions 7-10, choose the best answer. There is only one correct answer but you may choose up to *two*. If you choose two and one of the answers is correct, you will receive 1 mark.

7. (3 marks) If $y = \tan^{-1}(x)$, then $\frac{dy}{dx}$ is

A. $\frac{1}{\sqrt{1-x^2}}$

B. $\frac{1}{1+x^2}$

C. $\frac{1}{\sqrt{1+x^2}}$

D. $\frac{1}{1-x^2}$

8. (3 marks) If $f(x) = \frac{1}{\sqrt{1-x^2}}$, then $\int f(x)\,dx$ is

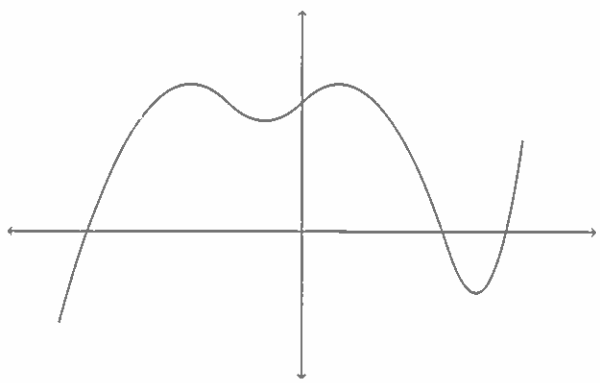
A. $\tan^{-1}(x) + C$

B. $\cos^{-1}(x) + C$

C. $\sin^{-1}(x) + C$

D. $\sec^{-1}(x) + C$

9. (3 marks) The graph of a function $f(x)$ is given below. How many critical points does $f(x)$ have?



A. 2

B. 3

C. 4

D. 5

10. (3 marks) Calculate $F(x) = \int 6x + 5\,dx$ if $F(2) = 14$.

A. $3x^2 + 5x - 6$

B. $3x^2 + 5x - 8$

C. $3x^2 + 5x - 10$

D. $3x^2 + 5x - 12$

For questions 11 − 21 you must show *all necessary work* to receive credit. Simply stating the answer will not be sufficient.

11. Consider $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.

(a) (2 marks) Determine whether or not $|x|$ is continuous at $x = 0$.

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} -x = -\lim_{x \to 0^-} x = -0 = 0$$

Both sides agree and equal $|0|$

$\Longrightarrow$ $|x|$ is continuous at 0

(b) (2 marks) Determine whether or not $|x|$ is differentiable at $x = 0$.

$$\lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

$$\lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1$$

limits do not agree $\Longrightarrow$ limit does not exist

$\Longrightarrow$ $|x|$ is not differentiable at 0.

12. (4 marks) Let $f(x) = 8x + 17$. Use the ε-δ definition of a limit to find $\lim_{x\to 3} f(x)$.

$$|F(x) - L| < \varepsilon \qquad 0 < |x - c| < \delta$$

$$|8x + 17 - 41| < \varepsilon \qquad 0 < |x - 3| < \delta$$

$$|8x - 24| < \varepsilon$$

$$8|x - 3| < \varepsilon \qquad \text{let } \delta = \varepsilon/8. \text{ Thus } \lim_{x\to 3} F(x) = 41$$

$$|x - 3| < \varepsilon/8$$

13. (5 marks) Find $\dfrac{dy}{dx}$ if $x^2y + \ln(y) = xy + 17 + \cos(x)$.

$$2xy + x^2\frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx} = y + x\frac{dy}{dx} - \sin(x)$$

$$x^2\frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx} - x\frac{dy}{dx} = y - \sin(x) - 2xy$$

$$\left(x^2 + 1/y - x\right)\frac{dy}{dx} = y - \sin(x) - 2xy$$

$$\boxed{\frac{dy}{dx} = \frac{y - \sin(x) - 2xy}{x^2 + 1/y - x}}$$

14. (5 marks) Find the volume of the solid formed by revolving the curve $y = \dfrac{2}{\sqrt{1+x^2}}$ about the x-axis over the interval $x = 0$ to $x = 1$.

$$V = \int_a^b \pi y^2 dx = \int_0^1 \pi \left(\frac{2}{\sqrt{1+x^2}}\right)^2 dx = 4\pi \int_0^1 \frac{1}{1+x^2} dx$$

$$= 4\pi \tan^{-1}(x) \Big|_0^1 = 4\pi [\pi/4 - 0] = \boxed{\pi^2}$$

15. (6 marks) Let $f(x) = x^2 + 3x + 5$. Find a formula for the Riemann sum of $f(x)$ obtained by dividing the interval $[1, 2]$ into n equal subintervals.

$$\frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$A = \sum_{j=1}^{n} f\left(a + \frac{b-a}{n} j\right) \frac{b-a}{n} = \frac{1}{n} \sum_{j=1}^{n} (1 + \tfrac{1}{n} j)^2 + 3(1 + \tfrac{1}{n} j) + 5$$

$$= \frac{1}{n} \sum_{j=1}^{n} 1 + \tfrac{2}{n} j + \tfrac{1}{n^2} j^2 + 3 + \tfrac{3}{n} j + 5$$

$$= \frac{1}{n} \sum_{j=1}^{n} 9 + \tfrac{5}{n} j + \tfrac{1}{n^2} j^2 = \frac{1}{n} \sum_{j=1}^{n} 9 + \frac{1}{n} \sum_{j=1}^{n} \tfrac{5}{n} j + \frac{1}{n} \sum_{j=1}^{n} \frac{1}{n^2} j^2$$

$$= \frac{9}{n} \sum_{j=1}^{n} 1 + \frac{5}{n^2} \sum_{j=1}^{n} j + \frac{1}{n^3} \sum_{j=1}^{n} j^2$$

$$= \frac{9}{n} \cdot n + \frac{5}{n^2} \cdot \frac{n(n+1)}{2} + \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \boxed{9 + \frac{5}{2}(1 + \tfrac{1}{n}) + \frac{1}{6}(2 + \tfrac{3}{n} + \tfrac{1}{n^2})}$$

16. (6 marks) Find the limit of $f(x) = \dfrac{\sqrt{x+1}-2}{x^2-9}$ as x approaches 3.

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x^2-9} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x\to 3} \frac{x+1-4}{(x^2-9)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{x-3}{(x-3)(x+3)(\sqrt{x+1}+2)} = \lim_{x\to 3} \frac{1}{(x+3)(\sqrt{x+1}+2)}$$

$$= \frac{1}{(3+3)(2+2)} = \boxed{\frac{1}{24}}$$

17. (6 marks) Explain why the function $f(x) = 2\cos(x) - 5x - 1$ has exactly one real root.

$f(0) = 1 > 0 \qquad f(\pi/2) = \frac{-5\pi}{2} - 1 < 0$

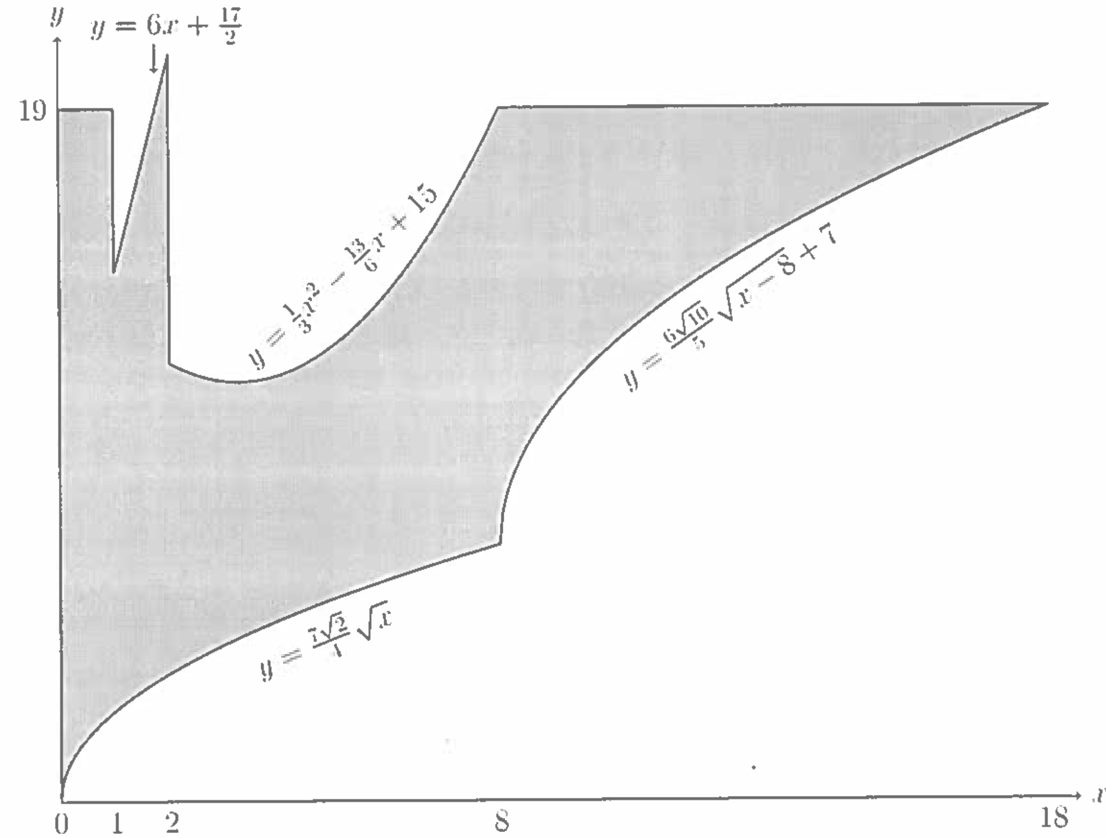
IVT $\Longrightarrow$ root in $(0, \pi/2)$

$f'(x) = -2\sin(x) - 5 < 0$ for all x

$\Longrightarrow$ $f(x)$ is strictly decreasing

$\Longrightarrow$ $f(x)$ has exactly one real root.

18. (a) (6 marks) Determine the area of the shaded region using the table provided.



$\int_1^2 6x + \frac{17}{2}\,dx = \frac{35}{2}$	$\int_2^8 \frac{1}{3}x^2 - \frac{13}{6}x + 15\,dx = 81$	$\int_8^{18} \frac{6\sqrt{10}}{5}\sqrt{x-8}\,dx = 80$	$\int_0^8 \frac{7\sqrt{2}}{4}\sqrt{x}\,dx = \frac{112}{3}$

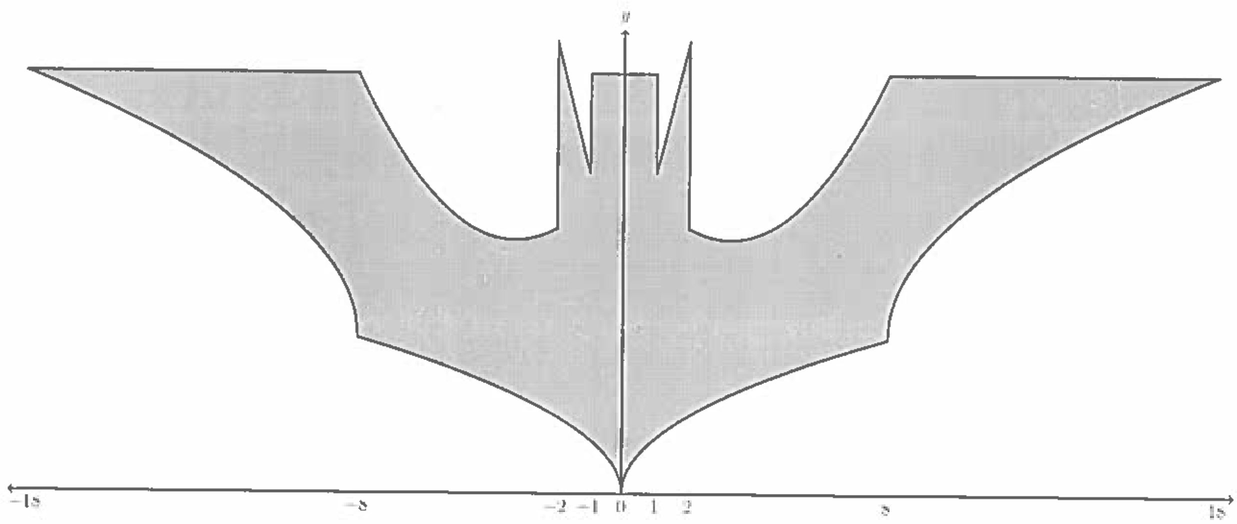
$$A = \int_0^1 19\,dx + \int_1^2 6x + \frac{17}{2}\,dx + \int_2^8 \frac{1}{3}x^2 - \frac{13}{6}x + 15\,dx$$

$$+ \int_8^{18} 19\,dx - \int_8^{18} \frac{6\sqrt{10}}{5}\sqrt{x-8}\,dx - \int_0^8 \frac{7\sqrt{2}}{4}\sqrt{x}\,dx$$

$$= 19 + \frac{35}{2} + 81 + 190 - 80 - \frac{112}{3}$$

$$= 210 + \frac{105}{6} - \frac{224}{6} = \frac{1260}{6} - \frac{119}{6} = \boxed{\frac{1141}{6}}$$

(b) (1 mark) Use your answer from part (a) to determine the area of the shaded region below.



$$A = 2\,\frac{1141}{6} = \boxed{\frac{1141}{3}}$$

19. (7 marks) Evaluate $\int_0^{\pi/4} 8\sin(x)\cos(x)\left(\cos^2(x)-\sin^2(x)\right)\,dx$.

$$= \int_0^{\pi/4} 4\sin(2x)\cos(2x)\,dx$$

$$= \int_0^{\pi/4} 2\sin(4x)\,dx$$

$$= -\frac{1}{2}\cos(4x)\Big|_0^{\pi/4} = -\frac{1}{2}[-1-1] = \boxed{1}$$

or

$$= \int_0^{\pi/4} 4\sin(2x)\cos(2x)\,dx \qquad \begin{aligned} u &= \sin(2x) \\ du &= 2\cos(2x)\,dx \end{aligned}$$

$$= \int_{x=0}^{x=\pi/4} 2u\,du = u^2\Big|_{x=0}^{x=\pi/4} = \sin^2(2x)\Big|_0^{\pi/4} = 1-0 = \boxed{1}$$

or

$$=8\int_0^{\pi/4} \sin(x)\cos^3(x)\,dx - 8\int_0^{\pi/4} \cos(x)\sin^3(x)\,dx$$

$$\begin{aligned} u &= \cos(x) \\ -du &= \sin(x)\,dx \end{aligned} \qquad \begin{aligned} v &= \sin(x) \\ dv &= \cos(x)\,dx \end{aligned}$$

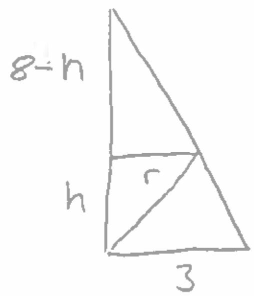
$$= -8\int_{x=0}^{x=\pi/4} u^3\,du - 8\int_{x=0}^{x=\pi/4} v^3\,dv = -2u^4 - 2v^4\Big|_{x=0}^{x=\pi/4}$$

$$= -2\cos^4(x) - 2\sin^4(x)\Big|_0^{\pi/4} = -2\left[\cos^4(\pi/4) + \sin^4(\pi/4) - (1+0)\right]$$

$$= -2\left[\left(\frac{\sqrt{2}}{2}\right)^4 + \left(\frac{\sqrt{2}}{2}\right)^4 - 1\right] = -4\left(\frac{\sqrt{2}}{2}\right)^4 + 2 = -4\left(\frac{1}{2}\right)^2 + 2$$

$$= -4\left(\frac{1}{4}\right) + 2 = -1 + 2 = \boxed{1}$$

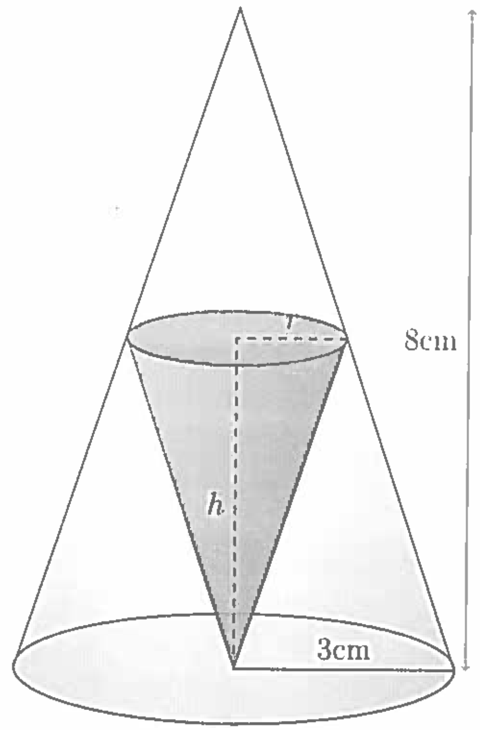
20. (10 marks) Determine the maximal volume of the cone that can be inscribed inside of a cone of radius 3cm and height 8cm, as shown below.



$$\frac{8-h}{r} = \frac{8}{3}$$

$$8-h = \frac{8}{3}r$$

$$\frac{8}{3}(3-r) = h$$



$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \frac{8}{3}(3-r)$$

$$= \frac{8\pi}{9}(3r^2 - r^3)$$

$$\frac{dV}{dr} = \frac{8\pi}{9}(6r - 3r^2) = \frac{8\pi}{3} r(2-r) = 0$$

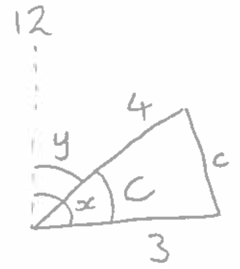
$$\Rightarrow r = 2$$

$$\Rightarrow V = \frac{8\pi}{9}(3(2)^2 - (2)^3)$$

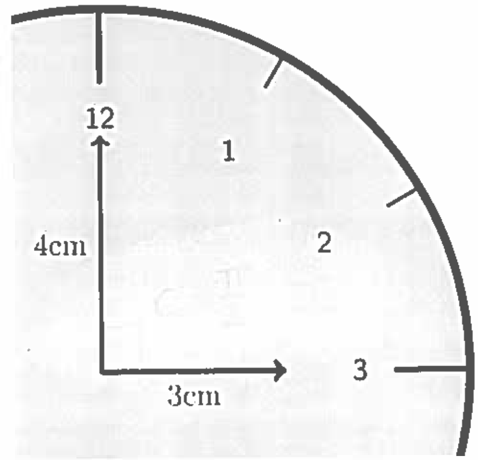
$$= \frac{8\pi}{9}(12 - 8)$$

$$= \boxed{\frac{32\pi}{9}\,cm^3}$$

21. (10 marks) Suppose that the minute hand of a clock is 4cm long and the hour hand is 3cm. How fast is the distance between the hour hand and the minute hand changing at 3pm? (*Hint: Use law of cosines*)



$t = 3$ hrs.



$$C = x - y = \frac{\pi}{6}t - 2\pi t$$
$$= -\frac{11}{6}\pi t$$

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$
$$= 4^2 + 3^2 - 2(4)(3)\cos\left(-\frac{11}{6}\pi t\right)$$
$$= 25 - 24\cos\left(-\frac{11}{6}\pi t\right)$$

$$c(3) = \sqrt{25 - 24\cos\left(-\frac{11}{2}\pi\right)}$$
$$= \sqrt{25 - 24\cos\left(\frac{1}{2}\pi\right.}$$
$$= 5$$

$$2c\frac{dc}{dt} = 24\sin\left(-\frac{11}{6}\pi t\right)\cdot -\frac{11}{6}\pi$$
$$= -44\pi\sin\left(-\frac{11}{6}\pi t\right)$$

$$\Rightarrow \frac{dc}{dt} = \frac{-22\pi}{c}\sin\left(-\frac{11}{6}\pi t\right)$$

$$\frac{dc}{dt}(3) = \frac{-22\pi}{5}\sin\left(-\frac{11}{2}\pi\right)$$
$$= \frac{-22\pi}{5}\sin\left(\frac{1}{2}\pi\right)$$
$$= \boxed{-\frac{22\pi}{5}\text{ cm/hr}}$$